Learning Goals Lecture C: Power series Representations of Functions

- Power Series representation of $\dfrac{1}{1-x}$.
- Master method of substitution.
- Definition of the Center of a power series, the Radius of Convergence and the Interval of Convergence.
- Master the techniques of integration and differentiation of power series.
- Be aware that the status of convergence at the end points of the Interval of Convergence may change with differentiation and integration.
- Become familiar with power series representations of commonly used functions such as

  $$\frac{1}{1-x}, \quad \frac{1}{1+x^k}, \qquad \ln(1+x), \qquad \arctan(x),$$

  and know how to derive them.
- Become familiar with common applications of power series such as approximating functions with polynomials, summing series, substituting polynomial approximations for functions in calculations.

Lecture C: Power series Representations of Functions, Stewart Section 11.9/11.8

In this section we are going to skip to sections 11.8/11.9 (mostly 11.9) in Stewart. We will go through the methods of determining whether a series converges or diverges as needed.

Recall that in the previous section, we saw that

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if} \quad -1 < r < 1$$

and this series diverges for $|r| \geq 1$.
If $a = 1$ this gives us that

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots \quad = \sum_{n=0}^{\infty} r^n \quad \text{for} \quad |r| < 1.$$

Now if we change the variable from r to x in the above equation, we get two formulas for the function $g(x) = \dfrac{1}{1-x}$ on the interval $-1 < x < 1$.

$$g(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1.$$

A formula for a function in the form of an infinite series such as the one shown on the right hand side above is called a power series representation for the function. We will see in this and future lectures that some familiar functions have a power series representation on part or all of their domain. A power series looks like a polynomial with infinitely many terms and we will see below that differentiating and integrating a power series is as easy as differentiating and integrating a polynomial. Also once we have a power series representation for a function, the first N terms give a polynomial of degree N with which we can estimate the values of the function. In a later lecture, we will discuss a theorem which shows us how to control the error of our estimates and thus we are free to choose a value of N which gives us an estimate with the desired level of accuracy. Because of these nice properties of power series, it is very desirable to have a power series representation for a function and you will see power series representations for functions used in many courses where calculus is applied.

Definition A **Power Series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where x is a variable, the c_n's are constants called the <u>coefficients</u> of the series.

Example In our example above $\sum_{n=0}^{\infty} x^n$ is a power series. The coefficients are all equal to 1, i.e. $c_0 = c_1 = c_2 = \cdots = 1$.

Example

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \ldots$$

Here $c_0 = 1/2^0 = 1, c_1 = 1/2^1, c_2 = 1/2^2, \ldots, c_n = 1/2^n$.

The Method of Substitution

We can use our power series representation for the function $g(x) = \dfrac{1}{1-x}$ above to derive power series representations for other functions using the technique of substitution (which amounts to a change of variable much like what we use for the method of substitution in integration).

Example (Substitution) Find a power series representation of the functions given below and find the interval of convergence of the series.

$$f(x) = \frac{1}{1+x},$$

Solution From above we have

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots = \sum_{n=0}^{\infty} y^n \quad \text{for} \quad -1 < y < 1$$

Now

$$\frac{1}{1+x} = \frac{1}{1-(-x)}.$$

Substituting $-x$ for y in the above equation, we get

$$\frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots = \sum_{n=0}^{\infty} (-x)^n \quad \text{for} \quad -1 < (-x) < 1$$

or

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{for} \quad -1 < x < 1}.$$

Since we have $-1 < -x < 1$ is equivalent to $1 > x > -1$ or $-1 < x < 1$, this power series representation for $f(x)$ remains valid on the interval $-1 < x < 1$.

Example Use the above method of substitution to find a power series representation for the function

$$f(x) = \frac{1}{1+x^7}$$

and find the interval on which this power series representation is valid.

Note Just as with polynomials, if I multiply a power series $\sum c_n x^n$ by a constant k, the result is the power series I get by multiplying each coefficient in the original power series by a constant, namely

$\sum kc_n x^n$. If I multiply a power series $\sum c_n x^n$ by x^m, I multiply each term of the power series by x^m to get a new power series $\sum c_n x^{n+m}$.

Example Find a power series representation for $\dfrac{2x}{1+x}$.

Solution We know from above that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{for} \quad -1 < x < 1.$$

Multiplying both sides of the above equation by $2x$, we get

$$\frac{2x}{1+x} = 2x - 2x^2 + 2x^3 - 2x^4 + \cdots = \sum_{n=0}^{\infty} (-1)^n 2x^{n+1} \quad \text{for} \quad -1 < x < 1.$$

Example Find a power series representation for $f_1(x) = \dfrac{3x^4}{1+5x^2}$ using a technique similar to that used in the above example along with substitution and find the interval on which this power series representation is valid.

The next example requires a little reshaping of the function before we apply the method of substitution:

Example Find a power series representation of the functions given below and find the interval on which this power series representation is valid.

$$g(x) = \frac{2x}{4+3x^3}$$

The following example is provided as an extra example at the end of your notes (click on the blue link to see the solution):

Example Find a power series representation of the function given below and find the interval on which this power series representation is valid.

$$g(x) = \frac{2x^2}{3-x},$$

Center of a Power Series

Example Notice that for the function $\frac{1}{4-x}$, we have two options for substitution.

A. We can rewrite the function as $\frac{1}{4}\left(\frac{1}{1-\left(\frac{x}{4}\right)}\right)$ and use the substitution $y = \frac{x}{4}$ as above,

or

B. we can rewrite the function as as $\frac{1}{1-(x-3)}$ and use our power series representation of $\frac{1}{1-y}$ from above with $y = (x-3)$.

The latter substitution gives us a power series representation for $\frac{1}{4-x}$ where the terms of the series are of the form $c_n(x-3)^n$. This type of power series is called a power series centered at 3. We get

$$\frac{1}{1-(x-3)} = 1 + (x-3) + (x-3)^2 + \cdots = \sum_{n=0}^{\infty}(x-3)^n \quad \text{when} -1 < x-3 < 1.$$

The representation holds true when $-1 < x-3 < 1$, adding 3 to both sides (of both inequalities) we get that the representation holds true when $2 < x < 4$. Notice that this interval is centered at 3 which coincides with the center of our power series.

Definition A power series in $(x-a)$ or **a power series centered at** a is a power series of the form

$$\sum_{x=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

where c_n is a constant for all n. The center of the power series is a.

Note that when $x = a$, we have

$$\sum_{x=0}^{\infty} c_n(x-a)^n = c_0 + c_1(a-a) + c_2(a-a) + c_3(a-a) + \cdots = c_0$$

Note Our previous definition of a power series is just a special case of this more general one with $a = 0$. When $a = 0$, the power series about a above becomes a power series of the form

$$\sum_{x=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

similar to the power series in our original definition and the previous examples.

We are often interested in the values of a function $f(x)$ near a particular value of x say a. In this case it is better to work with a power series representation of f around a than a power series representation of f around 0 (if $a \neq 0$) since the partial sums of the power series representation around a will converge to the values of the function near a faster than those of the power series representation around 0.

Example Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+1)^3}.$$

The center of this series is 1.

A Power series defines a function Note that any given power series $\sum c_n x^(x-a)^n$ defines a function of x where the value of the function $f(x) = \sum c_n x^(x-a)^n$ is the sum of the series on the right if it exists. As with definitions of all functions, unless otherwise stated, it is implicit in the definition that **the domain of the function f above** consists of all values of x where the definition makes sense, that **is all values of x where the series on the right converges.**

Radius of Convergence (R.O.C.) and Interval of Convergence (I.O.C.).

Theorem For any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only 3 possibilities for the the values of x for which the series converges :

1. The series converges only when $x = a$.
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Definition The **Radius of convergence (R.O.C.)** of the power series
is the number R in case 3 above.
In case 1, the Radius of convergence is 0 and
in case 2, the Radius of convergence is ∞.

We see that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ always converges within some interval centered at a and diverges outside that interval. The **Interval of Convergence** of a power series is the interval that consists of all values of x for which the series converges.

- In case 1 above, the interval of convergence is a single point $\{a\}$.
- In case 2 above the interval of convergence is $(-\infty, \infty)$.
- In case 3 above the interval of convergence may be

$$(a-R, a+R), \quad [a-R, a+R), \quad (a-R, a+R], \quad [a-R, a+R].$$

Example Recall from last day that the geometric series $\sum_{n=0}^{\infty} x^n$ converges if $|x|$ is less than 1 and diverges otherwise. The center of this series is $a = 0$ and the radius of convergence is 1. The interval of convergence is $(-1, 1)$.

Developing the skills to find the domains of functions defined using power series will take some time. Over the course of the next few lectures, we will do the necessary groundwork to determine the radius of convergence of a power series but for now, we wish to turn to two new methods of finding power series representations for well known functions, namely that of differentiation and that of integration.

Differentiation and Integration of Power Series

We can differentiate and integrate power series term by term, just as we do with polynomials.

Theorem If the series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$

Also

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$$

The radii of convergence of both of these power series is also R. (The interval of convergence may not remain the same when a series is differentiated or integrated; in particular convergence or divergence may change at the end points).

Example Find a power series representation of the function

$$\frac{1}{(1-x)^2}.$$

The following example is provided as an extra example at the end of your notes (click on the blue link to see the solution):

Example Find a power series representation of the function

$$\frac{1}{(x+1)^2}.$$

Example (Integration) Find a power series representation of the function

$$\ln(1+x).$$

Solution: Again we use the representation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n \quad \text{for} \quad -1 < x < 1$$

We have

$$\int \frac{1}{1+x}dx = \int [1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots]dx = \int \sum_{n=0}^{\infty}(-1)^n x^n dx \quad \text{for} \quad -1 < x < 1$$

integrating the left hand side and integrating the right hand side term by term, we get

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots = \sum_{n=0}^{\infty} \int (-1)^n x^n dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} + C$$

for $-1 < x < 1$. To find the appropriate constant term, we let $x = 0$ in this equation. We get

$$\ln(1+0) = C + 0 - 0 + 0 - 0 + \cdots = C$$

Therefore $C = 0$ and

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} \quad \text{for} \quad -1 < x < 1}$$

Example (Integration) Find a power series representation of the function

$$\arctan(x).$$

Summary and Application to Summing series

We will summarize some of the power series representations we have derived above in a table and use the table. This table is incomplete in that we do not know whether some of the power series representations converge at the end points of the interval of convergence. Although the radius of convergence of a power series remains the same when we integrate or differentiate, the convergence status of the end points of the interval of convergence can change. In the table below, a question mark above a "$<$" symbol below indicates that we are still uncertain as to whether to put a "$<$" symbol or a "$\leq$" symbol there. We will expand this table as we go and we will fill in the missing information as we progress through the various results about power series and tests for convergence.

Note When $x = -1$, $\ln(1 + x)$ is undefined and also the corresponding series

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)}{n+1}$$

$$= -\left(1 + \frac{1}{2} + \frac{1}{3} + \dots\right) = -\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{which diverges.}$$

function	Power series Repesentation	Interval
$\dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\dfrac{1}{1+x^k}$	$\sum_{n=0}^{\infty} (-1)^n x^{kn}$	$-1 < x < 1$
$\ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{n+1}}{n+1}$	$-1 < x \overset{?}{<} 1$
$\arctan(x)$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	$-1 \overset{?}{<} x \overset{?}{<} 1$

Clearly this is just a small sample of the number of such representations that you can derive with these simple but powerful methods, so rather than memorizing power series representation for thousands of functions individually, knowing a few basic power series and how to use the above methods to derive new ones is very important.

Example Use the table above to find the sum of the following series

$$A. \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{3}^{2n+1} \cdot (2n+1)}, \qquad B. \quad \sum_{n=0}^{\infty} \frac{(-1)}{2^{n+1}(n+1)}$$

For A, we see that if we take $x = 1/\sqrt{3}$, we get

$$\pi/6 = \arctan(1/\sqrt{3}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{3}^{2n+1} \cdot (2n+1)}.$$

B (you try):

Application: Estimation Often in applications of calculus, where calculations of integrals and derivatives are difficult or impossible, one can get a good estimate of the answer by substituting the first few terms of the power series representation of a function.

Example Evaluate the integral

$$\int \frac{1}{1+x^{50}}\,dx.$$

This could theoretically be solved by hand using partial fractions, but that is a very long calculation. If we are only interested in the values of this function for values of x near 0 and if we are happy with an estimate of its value, then we can get a reasonable estimate by substituting the first few terms of the power series representation of $\frac{1}{1+x^{50}}$ in the calculation. The more terms I include, the better my approximation will be, we will see how to control this error in a later lecture. Let us include the first three terms to get the approximation

$$\int \frac{1}{1+x^{50}}\,dx \approx \int (1 - x^{50} + x^{100})\,dx = x - \frac{x^{51}}{51} + \frac{x^{101}}{101} + C.$$

Example Use the first three terms of the power series representation of $\frac{1}{1+x^7}$ to estimate

$$\int_0^{0.1} \frac{1}{1+x^7}\,dx.$$

Example Find a polynomial estimate for the following integral for values of x near 0:

$$\int \arctan(x)\ln(1+x)dx.$$

Extras

Extra Example Power Series Representation

Example Find a power series representation of the function given below and find the interval on which this power series representation is valid.

$$g(x) = \frac{2x^2}{3 - x},$$

We have

$$\frac{2x^2}{3-x} = \frac{2x^2}{3}\left[\frac{1}{1-(x/3)}\right].$$

Now recall from above that

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots + y^n + \cdots = \sum_{n=0}^{\infty} y^n \quad \text{for} \quad -1 < y < 1$$

Therefore, substituting $x/3$ for y, we get

$$\frac{1}{1-\left(\frac{x}{3}\right)} = 1 + \left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \cdots + \left(\frac{x}{3}\right)^n + \cdots = \sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \quad \text{for} \quad -1 < \left(\frac{x}{3}\right) < 1$$

We have $-1 < \left(\frac{x}{3}\right) < 1$ if $-3 < x < 3$ (multiplying the inequality by 3). Therefore

$$\frac{1}{1-\left(\frac{x}{3}\right)} = 1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots + \frac{x^n}{3^n} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{3^n} \quad \text{for} \quad -3 < x < 3.$$

Now we want a power series representation for

$$g(x) = \frac{2x^2}{3-x} = \frac{2x^2}{3}\left[\frac{1}{1-(x/3)}\right]$$

using the power series derived above for $\frac{1}{1-(x/3)}$, we get

$$\frac{2x^2}{3-x} = \frac{2x^2}{3}\left[1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots + \frac{x^n}{3^n} + \cdots\right] = \frac{2x^2}{3}\sum_{n=0}^{\infty}\frac{x^n}{3^n} \quad \text{for} \quad -3 < x < 3.$$

or

$$\frac{2x^2}{3-x} = \left[\frac{2x^2}{3} + \frac{2x^2}{3}\left(\frac{x}{3}\right) + \frac{2x^2}{3}\left(\frac{x^2}{3^2}\right) + \frac{2x^2}{3}\left(\frac{x^3}{3^3}\right) + \cdots + \frac{2x^2}{3}\left(\frac{x^n}{3^n}\right) + \cdots\right] = \sum_{n=0}^{\infty}\frac{2x^2}{3}\left(\frac{x^n}{3^n}\right) \quad \text{for} \ -3 < x < 3.$$

or

$$\frac{2x^2}{3-x} = \left[\frac{2x^2}{3} + \frac{2x^3}{3^2} + \frac{2x^4}{3^3} + \frac{2x^5}{3^4} + \cdots + \frac{2x^{n+2}}{3^{n+1}} + \cdots\right] = \sum_{n=0}^{\infty}\frac{2x^{n+2}}{3^{n+1}} \quad \text{for} \quad -3 < x < 3.$$

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Extra Example Power Series Differentiation

Example Find a power series representation of the function

$$\frac{1}{(x+1)^2}.$$

Above we found that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n \quad \text{for} \quad -1 < x < 1$$

Therefore we have

$$\frac{d}{dx}\left[\frac{1}{1+x}\right] = \frac{d}{dx}\Big[1 - x + x^2 - x^3 + \cdots\Big] = \frac{d}{dx}\Big[\sum_{n=0}^{\infty}(-1)^n x^n\Big] \quad \text{for} \quad -1 < x < 1$$

Differentiating we get

$$\frac{-1}{(1+x)^2} = 0 - 1 + 2x - 3x^2 + \cdots + (-1)^n n x^{n-1} + \cdots = \sum_{n=0}^{\infty}\frac{d}{dx}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n n x^{n-1} = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$$

when $-1 < x < 1$. Since the series has the same radius of convergence when differentiated, we know that this new series converges on the interval $-1 < x < 1$.
(Note we can set the limits of the new sum from n = 0 to infinity if we like, since that just gives an extra 0 at the beginning or we can drop the n=0 term; this is merely a cosmetic change.)
Now we multiply both sides by -1 to get

$$\boxed{\frac{1}{(1+x)^2} = 0 + 1 - 2x + 3x^2 + \cdots + (-1)^{n+1} n x^{n-1} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} \quad \text{for} \quad -1 < x < 1}$$

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Application from Probability When using probability one often deals with discrete random variables which have infinitely many possible values which can be listed in a certain order. For these variables calculation of the usual statistics such as the expected value and standard deviation involve summing an infinite series. Suppose, for example, an experiment consists of flipping a coin until we see a head and we wish to know the average number of flips for such an experiment.

The possible outcomes for such an experiment are infinite; $\{T, HT, HHT, \dots\}$. Using the fact that heads and tails both have probability 1/2 on each coin flip and that coin flips are independent (allowing us to multiply probabilities), we get the probability distribution for the experiment and the random variable X = number of coin flips shown below.

Value of X	**Outcome**	**Probability**
1	H	1/2
2	HT	1/4
3	HHT	1/8
4	HHHT	1/16
$\vdots$	$\vdots$	$\vdots$
n	$\underbrace{HH\dots H}_{\text{n - 1 times}}T$	$\frac{1}{2^n}$
$\vdots$	$\vdots$	$\vdots$

If we were to run this experiment many times, the average number of coin flips per trial of the experiment would be roughly equal to the expected value of X which is given by the sum of the values of X multiplied by their respective probabilities. That is, the average number of coin flips is

$$E(X) = \sum_{n=1}^{\infty} n\frac{1}{2^n}$$

Now we saw above that if $f(x) = \dfrac{1}{1-x}$, then

$$f(x) = \sum_{n=0}^{\infty} x^n \quad \text{for } -1 < x < 1 \text{ and} \qquad f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} \quad \text{for } -1 < x < 1$$

When we set $x = 1/2$ in the latter equation, we get

$$4 = \frac{1}{(1-(1/2))^2} = \sum_{n=1}^{\infty} n\frac{1}{2^{n-1}}.$$

Therefore

$$2 = \frac{1}{2}\cdot 4 = \sum_{n=1}^{\infty} n\frac{1}{2^n} = E(X).$$

Thus the average number of coin flips in such and experiment is 2.